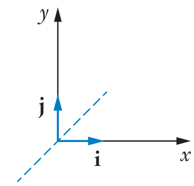
5.

Vectors in three dimensions

- Three-dimensional vectors
- The angle between two lines
- Vector product (cross product)
- Vector equation of a line
- Vector equation of a plane
- Given three points that lie in the plane
- Interception/collision
- Vector equation of a sphere
- Distance from a point to a line revisited
- Miscellaneous exercise five

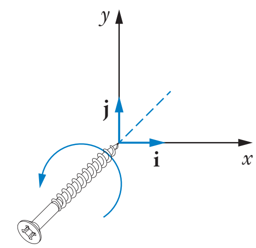
The section on vectors in the *Preliminary work* section only involved vectors in *two* dimensions, as did the vector work of chapter 4. The unit vectors **i** and **j** were used to express such two-dimensional vectors in component form.



We take **i** and **j** to be unit vectors in the direction of the positive x- and y-axes as shown on the right.

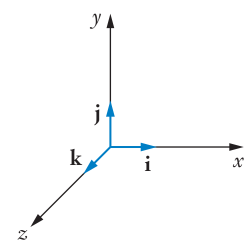
To consider vectors in three dimensions we need a third unit vector, **k**, perpendicular to **i** and **j** and acting along a z-axis. The positive direction of this z-axis will either be into, or out of, the page.

To determine which of these directions to choose we use the 'right hand screw' convention. If we imagine a normal screw at the origin, perpendicular to the x-y plane, being screwed from x to y then the direction in which the screw will move gives the positive direction of the z-axis.

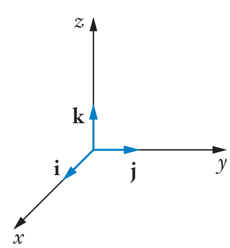


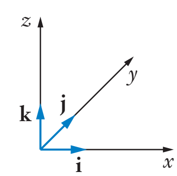
Alternatively, with your right hand, point your first finger (index finger) in the positive direction of the x-axis and your middle finger in the positive direction of the y-axis. Your thumb then gives the positive direction of the z-axis.

Thus with **i** and **j** as shown the third unit vector, **k**, will be as illustrated on the right, i.e. out of the page.



If we draw x and y axes differently then these right hand rules again allow the direction of the positive z-axis to be determined.





The vector ideas we have developed for vectors in two dimensions can simply be extended to three dimensions.

For example if $\quad \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \quad$ and $\quad \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \quad$ then:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + (5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) \\ &= 7\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{aligned} \qquad \begin{aligned} \mathbf{a} - \mathbf{b} &= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) \\ &= -3\mathbf{i} + 5\mathbf{j} - 11\mathbf{k} \end{aligned}$$

$$\begin{aligned} 2\mathbf{a} &= 2\,(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \\ &= 4\mathbf{i} + 6\mathbf{j} - 8\mathbf{k} \end{aligned} \qquad \begin{aligned} \mathbf{a}\,.\,\mathbf{b} &= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\,.\,(5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) \\ &= (2)(5) + (3)(-2) + (-4)(7) \\ &= -24 \end{aligned}$$

Which of the following will be $|\mathbf{a}|$:

$$\sqrt{(2)^2 + (3)^2 + (-4)^2} \qquad \text{or} \qquad \sqrt[3]{(2)^3 + (3)^3 + (-4)^3}\ ?$$

Attempt the situation on the next page.

Situation (Challenging)

It was the year 2100 and, as coincidence would have it, that was also the time of day. Space travel was commonplace. The person-made space station located at (0, 0, 0) was the space traffic control point. The locations of other space stations, planets and space vehicles were then given with respect to (0, 0, 0).

Victor Lim was on space traffic control duty on space station (0, 0, 0) and was in radio contact with three Space Buses A, B and C.

Space Bus A was piloted by Captain Hector.
Space Bus B was piloted by Captain Over.
Space Bus C was piloted by Captain Rover.

'What's my vector, Victor? Over.' said Hector.

'You're at $1000\mathbf{i} - 1500\mathbf{j} + 3000\mathbf{k}$, Hector' replied Victor. 'What's your velocity? Over.'

'Do you want it as a vector, Victor? Over.'

'Yes, please. A vector, Hector. Over.' said Victor, wondering how else Captain Hector thought he would give his velocity.

'It's $2500\mathbf{i} + 3000\mathbf{j} - 2000\mathbf{k}$, over.'

No sooner had Victor checked that the voice input device had correctly assimilated and confirmed the data contained in the conversation than the radio again crackled into life. It was Space Bus B and Captain Over.

'Hi Victor, it's Over, over?'

'What's over? Over.' replied Victor.

'Nothing's over, Victor, it's me Captain Over. I'm just confirming my velocity vector, Victor.
It's $-2700\mathbf{i} - 900\mathbf{j} - 1800\mathbf{k}$, can you confirm my position vector, Victor? Over.'

'It's $2200\mathbf{i} + 300\mathbf{j} + 6600\mathbf{k}$, Over. Over.'

'Thank you,' said Over. 'Over.'

'Is that you, Rover? Over.'

'Yes,' said Rover. 'My velocity vector is $3600\mathbf{i} + 1800\mathbf{j} - 450\mathbf{k}$, Victor. What's my position vector? Over.'

'It's $100\mathbf{i} - 600\mathbf{j} + 6150\mathbf{k}$. Over.'

'Thanks Victor. Over.'

Questions

Assume that all of the above vectors describe the situation at 2100 hours.
Assume also that distances are in kilometres and velocities in km/h.

1. How far was Hector from Victor at 2100 hours?
2. How far was Hector from Over at 2100 hours?
3. What was Rover's speed at 2100 hours?
4. On what vector from Hector is Rover at 2100 hours?
5. Show that unless Over and/or Rover change course they will collide and find the time this would occur and the position vector of the location.
6. What is the closest distance Hector comes to Victor and when does this closest distance occur? (Give distance to the nearest kilometre and time to the nearest minute.)
7. When is Hector closest to Over? (Answer to the nearest minute and assume it was **not** Over that changed course to avoid collision with Rover.)

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Three-dimensional vectors

The previous situation rather 'threw you in the deep end' with regard to vectors in three dimensions. However the techniques involved are the same as we used for similar situations involving two dimensions, we simply extend these ideas to involve the third dimension. Well done if you managed the situation but do not be too worried if you found it challenging, as it was not an easy situation.

Whether you managed the previous situation or not, read through the following examples as they show how the two-dimensional vector ideas you are already familiar with can be extended to three dimensions. Note especially examples 2 and 4 which use the column matrix form of representing a vector.

In this form of representation the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is written $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

You may find this column matrix a particularly useful form of representation when three dimensions are involved.

EXAMPLE 1

If $\mathbf{a} = 14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ find

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - \mathbf{b}$ **c** $|\mathbf{a}|$ **d** a unit vector parallel to **a**.

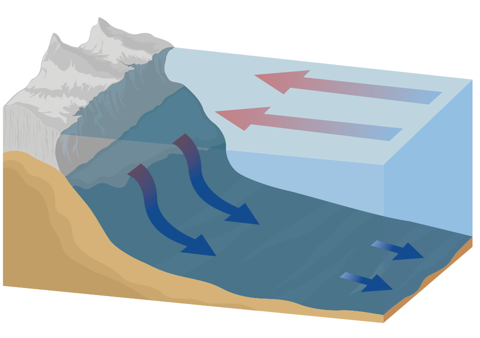
Solution

a
$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 15\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} \end{aligned}$$

b
$$\begin{aligned} 2\mathbf{a} - \mathbf{b} &= 2(14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= (28\mathbf{i} - 10\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 27\mathbf{i} - 12\mathbf{j} + \mathbf{k} \end{aligned}$$

c
$$\begin{aligned} |\mathbf{a}| &= |14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}| \\ &= \sqrt{(14)^2 + (-5)^2 + (2)^2} \\ &= 15 \end{aligned}$$

d A unit vector parallel to **a** is $\frac{1}{15}(14\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$.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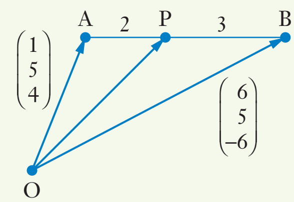
EXAMPLE 2

Point A has position vector $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ and point B has position vector $\begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix}$.

Find the position vector of the point P that divides AB internally in the ratio 2 : 3.

Solution

If P divides AB in the ratio 2 : 3 then $\overrightarrow{AP} : \overrightarrow{PB} = 2:3$, as shown in the diagram on the right.



$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \overrightarrow{OA} + \frac{2}{5}\overrightarrow{AB}$$

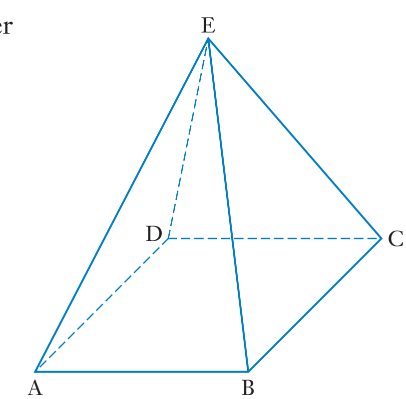
But $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$ and $\overrightarrow{AB} = -\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$

$$\therefore \quad \overrightarrow{OP} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

The point that divides AB internally in the ratio 2 : 3 has position vector $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$.

The angle between two lines

In two dimensions two lines that are not parallel must cut each other somewhere. This is not the case in three dimensions where two non-parallel lines may be such that they have no point in common. This is the case with the lines EA and BC in the square based pyramid ABCDE shown on the right. EA and BC do not intersect. They are said to be **skew** lines. Skew lines do not intersect and are not parallel.



We can still refer to the angle between skew lines but in this case we mean the angle between one of the skew lines and another line drawn parallel to the second skew line and intersecting the first.

Thus the angle between the skew lines EA and BC would be ∠EAD because AD is parallel to BC and does meet EA.

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EXAMPLE 3

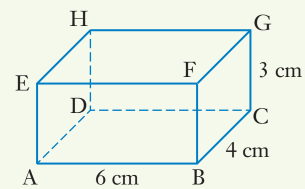
The diagram shows a rectangular prism ABCDEFGH.

AB = 6 cm, BC = 4 cm and CG = 3 cm.

Use vector techniques to find, in degrees and correct to one decimal place:

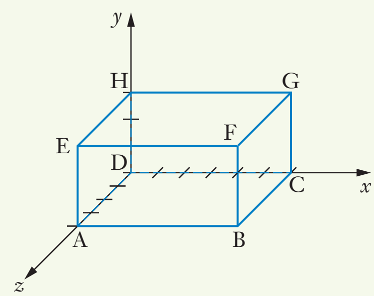
a $\angle$FDB,

b the acute angle between the skew lines DB and HE.



Solution

a Consider x, y and z-axes as shown on the right.



$$
\begin{aligned}
\text{Then} \quad \overrightarrow{DF} &= 6\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \\
\text{and} \quad \overrightarrow{DB} &= 6\mathbf{i} + 4\mathbf{k}. \\
\therefore \quad \overrightarrow{DF} \cdot \overrightarrow{DB} &= (6)(6) + (3)(0) + (4)(4) \\
&= 52 \\
\therefore \quad 52 &= \sqrt{6^2 + 3^2 + 4^2}\sqrt{6^2 + 0^2 + 4^2} \cos \angle FDB \\
\text{giving} \quad \angle FDB &\approx 22.6°
\end{aligned}
$$

b

$$
\begin{aligned}
\overrightarrow{DB} \cdot \overrightarrow{HE} &= (6\mathbf{i} + 4\mathbf{k}) \cdot (4\mathbf{k}) \\
&= 16 \\
\therefore \quad 16 &= \sqrt{6^2 + 4^2}(4)\cos\theta \text{ where } \theta \text{ is the required angle.} \\
\therefore \quad \theta &\approx 56.3°
\end{aligned}
$$

The acute angle between the skew lines DB and HE is approximately 56.3°.

EXAMPLE 4

If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{d} = 9\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ express $\mathbf{d}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$.

Solution

$$
\begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \eta\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}
$$

$$
\begin{aligned}
\text{Hence} \quad 9 &= 2\lambda + 3\mu + \eta \\
5 &= \lambda - 2\mu - 2\eta \\
2 &= -\lambda + 4\mu + \eta
\end{aligned}
$$

Solving simultaneously, with the assistance of a calculator, gives

$$\lambda = 3, \mu = 2 \text{ and } \eta = -3.$$

Thus $\quad \mathbf{d} = 3\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}.$

$$\begin{cases} 9 = 2\lambda + 3\mu + \eta \\ 5 = \lambda - 2\mu - 2\eta \\ 2 = -\lambda + 4\mu + \eta \end{cases} \Big|\, \lambda, \mu, \eta$$

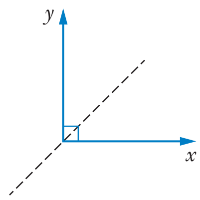
$$\{\lambda = 3, \mu = 2, \eta = -3\}$$

Note: Chapter Six, *Systems of linear equations*, will consider the solution of three equations in three unknowns in more detail. For now, use your calculator to solve the equations simultaneously.

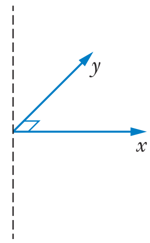
Exercise 5A

1 Copy each of the following and indicate on your drawing the direction of the positive z-axis.

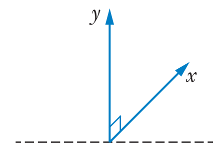
a



b



c



2 If $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ find

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $2\mathbf{a} + \mathbf{b}$ **d** $2(\mathbf{a} + \mathbf{b})$

e $\mathbf{a}.\mathbf{b}$ **f** $\mathbf{b}.\mathbf{a}$ **g** $|\mathbf{a}|$ **h** $|\mathbf{a} + \mathbf{b}|$

3 If $\mathbf{c} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ find

a $\mathbf{c} + \mathbf{d}$ **b** $\mathbf{c} - \mathbf{d}$ **c** $2\mathbf{c} + \mathbf{d}$ **d** $2(\mathbf{c} + \mathbf{d})$

e $\mathbf{c}.\mathbf{d}$ **f** $\mathbf{d}.\mathbf{c}$ **g** $|\mathbf{c}|$ **h** $|\mathbf{c} + \mathbf{d}|$

4 If $\mathbf{e} = <1, 4, -3>$ and $\mathbf{f} = <-1, 2, 0>$ find

a $\mathbf{e} - \mathbf{f}$ **b** $\mathbf{e} - 2\mathbf{f}$ **c** $2\mathbf{e} + \mathbf{f}$ **d** $\mathbf{e} + \mathbf{f}$

e $\mathbf{e}.\mathbf{f}$ **f** $(2\mathbf{e}).(3\mathbf{f})$ **g** $(\mathbf{e} - \mathbf{f}).(\mathbf{e} - \mathbf{f})$ **h** $|\mathbf{e} - \mathbf{f}|$

5 A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $-5\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. Express the following in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{CA}$ **d** $\overrightarrow{AC}$

6 If $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$ find

a $\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} + \mathbf{r}$ **c** $(\mathbf{p} + \mathbf{q}).(\mathbf{q} + \mathbf{r})$

7 If $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 14\mathbf{j} + 5\mathbf{k}$ find

a $|\mathbf{u}|$ **b** $|\mathbf{v}|$ **c** $\mathbf{u}.\mathbf{v}$

d the angle between $\mathbf{u}$ and $\mathbf{v}$.

8 Points A and B have position vectors $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ respectively, with respect to an origin O. Find the size of $\angle AOB$ correct to the nearest degree.

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9 Find the angle between $\mathbf{p} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, giving your answer correct to the nearest degree.

10 Find the angle between $\mathbf{s} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, correct to the nearest degree.

11 If $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{s} = 3\mathbf{i} + 4\mathbf{k}$ find

a a unit vector in the same direction as **r**,

b a vector in the same direction as **r** but equal in magnitude to **s**,

c a vector in the same direction as **s** but equal in magnitude to **r**,

d the angle between **r** and **s** (to the nearest degree).

12 For each of the following, state whether the given pair of vectors are parallel, perpendicular or neither of these.

a $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$

b $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

c $\langle 1, 3, -2 \rangle$ and $\langle -2, 3, 1 \rangle$

d $\langle 1, 2, 3 \rangle$ and $\langle 3, 3, -3 \rangle$

e $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$

f $\begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

g $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$

13 Find the magnitude of the resultant of the three forces

$\mathbf{F}_1 = (5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k})$ N, $\mathbf{F}_2 = (3\mathbf{i} + 5\mathbf{j} + 5\mathbf{k})$ N and $\mathbf{F}_3 = (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ N.

14 Point A has position vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{BA} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Find the position vector of B.

15 Find vectors **a** and **b** such that $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$.

16 If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 6 \\ p \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 7 \\ q \\ -2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 3 \\ -4 \\ r \end{pmatrix}$ find p, q and r given that:

b is parallel to **a**,

c is perpendicular to **a**,

d is perpendicular to **b**.

17 A particle has an initial position vector of $(-4\mathbf{i} - 4\mathbf{j} + 11\mathbf{k})$ m with respect to an origin O. The particle moves with constant velocity of $(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ m/s. What will be the position vector of the particle after

a 1 second?

b 2 seconds?

c How far will the particle be from O after 3 seconds?

d After how many seconds will the particle be 15 metres from O?

18 Points A, B and C have position vectors $7\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $2\mathbf{i} - 5\mathbf{k}$ respectively. Prove that A, B and C are collinear.

19 Points A and B have position vectors $\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 9 \\ -1 \end{pmatrix}$ respectively. Find the position vector of the point that divides AB internally in the ratio 2 : 3.

20 Points A and B have position vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $4\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. Find the position vector of the point P if $\overrightarrow{AB} = \overrightarrow{BP}$.

21 A and B have position vectors $5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $9\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$ respectively. Find the position vector of the point P if $\overrightarrow{AP} : \overrightarrow{AB} = 3 : 4$.

22 Points A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $3\mathbf{k}$ and $4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ respectively. Prove that △ABC is right-angled.

23 Find the acute angles that the vector $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ makes with the x-, y- and z-axes. (Hint: Consider $\mathbf{a}\,.\,\mathbf{i}$.)

24 If $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ express each of the following in the form $\lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c}$.

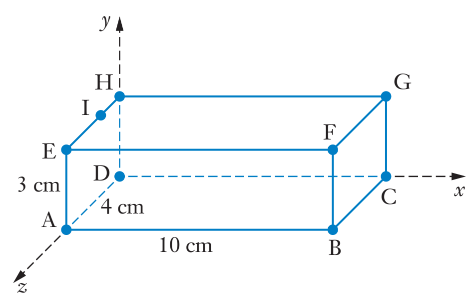
$\mathbf{d} = 7\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$, $\mathbf{e} = \mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$, $\mathbf{f} = 2\mathbf{j} - 2\mathbf{k}$.

25 A rectangular block ABCDEFGH is placed with DC along the x-axis, DA along the z-axis and DH along the y-axis (see diagram).

AB = 10 cm, BC = 4 cm and AE = 3 cm.

I is on HE and HI = 1 cm.

The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are along the x-, y- and z-axes respectively.



a Find $\overrightarrow{DC}$, $\overrightarrow{DB}$ and $\overrightarrow{DI}$ in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

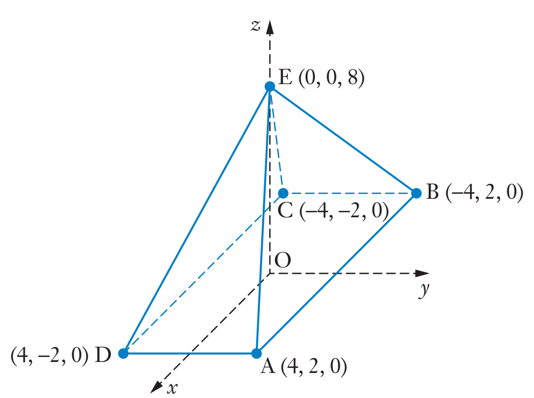
b Use vector techniques to determine ∠IDB to the nearest degree.

26 The right pyramid ABCDE is shown on the right with the coordinates of the vertices as indicated.

Use vector techniques to determine

a ∠OAE,

b the acute angle between the skew lines AE and DB.



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27 Points A, B and C have position vectors $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$ respectively.

a Find $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.

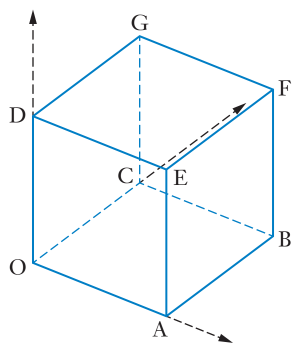
b Prove that $\triangle ABC$ is isosceles.

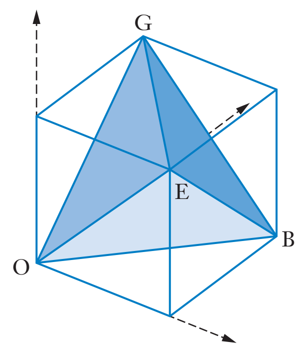
c Find $\overrightarrow{AC} \cdot \overrightarrow{AC}$

d Find the angles of the triangle.

28 The diagram below left shows a cube OABCDEFG with

$\overrightarrow{OA} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, $\overrightarrow{OC} = 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$, $\overrightarrow{OD} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}$.



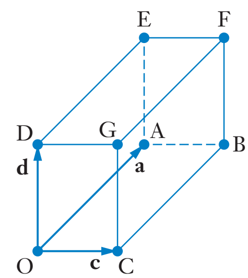


Prove that OBGE is a regular tetrahedron. (A regular tetrahedron has four congruent equilateral triangular faces.)

29 The diagram, right, shows a rectangular prism OABCDEFG with

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, and $\overrightarrow{OD} = \mathbf{d}$.

Prove that the diagonals OF, AG, BD and CE intersect at their mid-points.



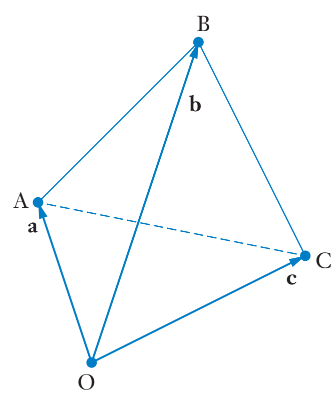
30 The diagram, right, shows a tetrahedron OABC with

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{OC} = \mathbf{c}$.

The tetrahedron has three pairs of opposite edges:

OC and BA, OB and CA, OA and CB.

Prove that if any two of these pairs involve perpendicular sides, e.g. OC perpendicular to BA and OB perpendicular to CA, then the third pair also involves perpendicular sides, i.e. OA is perpendicular to CB.



Vector product (cross product)

The point was made in the book for Unit One of the *Mathematics Specialist* course that the idea of forming a product of two vectors may initially seem rather confusing. How do we multiply together quantities which have magnitude and direction? Whilst we could define what we mean by vector multiplication in all sorts of ways there are two methods of performing vector multiplication that prove to be useful. One method of vector multiplication gives an answer that is a scalar. This is the **scalar product**, a concept we are already familiar with. A second method gives an answer that is a vector. We call this the **vector product**, a concept that we will consider now.

For vectors **a** and **b** the vector product is written $\mathbf{a} \times \mathbf{b}$, is also referred to as the *cross product* and is a vector perpendicular to both **a** and **b**. We say that $\mathbf{a} \times \mathbf{b}$ is a vector **normal** to the plane containing **a** and **b**.

Suppose, for example, that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$,
and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Using a calculator $\mathbf{a} \times \mathbf{b} = -\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}$

$$\text{crossP}\left(\begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}\right) \qquad \begin{bmatrix} -1 \\ -11 \\ -7 \end{bmatrix}$$

The working below confirms that this vector is perpendicular to both **a** and **b**:

$$\begin{aligned} (2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) \cdot (-\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}) &= -2 - 33 + 35 \\ &= 0 \end{aligned}$$

$$\begin{aligned} (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}) &= -1 + 22 - 21 \\ &= 0 \end{aligned}$$

The vector product of two vectors can be determined from the **i**-**j**-**k** components as follows.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

then $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

This formula may appear complicated but if we write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ to represent $ad - bc$ (which you may recognise as the determinant of the 2×2 matrix) then:

$$\begin{array}{ccccc} \mathbf{i} & & \mathbf{j} & & \mathbf{k} \\ \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} & & \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} & & \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \\ \downarrow & & \downarrow & & \downarrow \\ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & + & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{array}$$

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} \quad - \quad (a_1b_3 - a_3b_1)\mathbf{j} \quad + \quad (a_1b_2 - a_2b_1)\mathbf{k}$$

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EXAMPLE 5

With $\mathbf{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{d} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ determine $\mathbf{c} \times \mathbf{d}$ and confirm that your answer is indeed a vector that is perpendicular to both **c** and **d**.

Solution

$$\begin{matrix} 4 & -1 & 3 \\ -1 & 2 & -1 \end{matrix}$$

$$\begin{aligned}
\mathbf{c} \times \mathbf{d} &= (1-6)\mathbf{i} - (-4+3)\mathbf{j} + (8-1)\mathbf{k} \\
&= -5\mathbf{i} + \mathbf{j} + 7\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{c}.(-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) &= (4\mathbf{i} - \mathbf{j} + 3\mathbf{k}).(-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \\
&= -20 - 1 + 21 \\
&= 0
\end{aligned}$$

$-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ is perpendicular to **c**.

$$\begin{aligned}
\mathbf{d}.(-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) &= (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}).(-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \\
&= 5 + 2 - 7 \\
&= 0
\end{aligned}$$

$-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ is perpendicular to **d**.

Note

- The syllabus for this unit, at the time of writing, says that students studying the unit should be able to '*use the cross product to determine a vector normal to a given plane*'. Whilst we will concentrate on such use in this book it is also worth noting that just as the scalar product, in addition to being determinable from the **i**-**j**-**k** components, also has the meaning

  $$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, \text{ where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b},$$

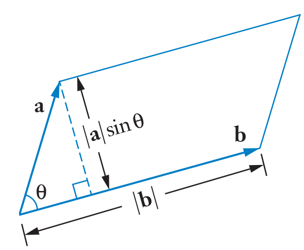
  then so, not proved here,

  $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta, \text{ where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}.$$

  If $\hat{\mathbf{n}}$ is a unit vector (i.e. a vector of unit length) in the direction of $\mathbf{a} \times \mathbf{b}$ then this could be written

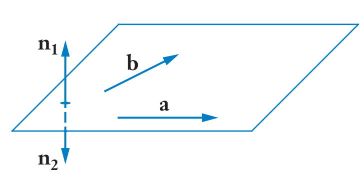
  $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}.$$

- The previous dot point means that we can interpret $|\mathbf{a} \times \mathbf{b}|$ as being the area of a parallelogram that has **a** and **b** as adjacent sides.



- There are two possible directions for a vector perpendicular to the plane containing vector **a** and **b**, as the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ indicate in the diagram on the right. To determine which of these is the direction of $\mathbf{a} \times \mathbf{b}$ we again use the right hand screw rule. If we rotate a normal screw from **a** to **b**, the direction the screw would move tells us the direction of $\mathbf{a} \times \mathbf{b}$. In this case the direction of $\mathbf{n}_1$.

  $\mathbf{b} \times \mathbf{a}$ would be in the direction of $\mathbf{n}_2$.



Exercise 5B

1 The fact that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ means that two parallel vectors should have a cross product equal to the zero vector.

Prove that the method for determining the cross product of two vectors from their **i**-**j**-**k** components also gives the zero vector if the two vectors are parallel.

2 With $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ determine the vector product $\mathbf{a} \times \mathbf{b}$ and confirm that your answer is indeed a vector that is perpendicular to both **a** and **b**.

3 With $\mathbf{c} = 5\mathbf{i} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ determine the vector product $\mathbf{c} \times \mathbf{d}$ and confirm that your answer is indeed a vector that is perpendicular to both **c** and **d**.

4 With $\mathbf{p} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = -\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ determine the vector product $\mathbf{p} \times \mathbf{q}$ and confirm that your answer is indeed a vector that is perpendicular to both **p** and **q**.

5 Without applying the formula, but just by applying some thought, what would you expect $\mathbf{a} \times \mathbf{b}$ to equal if $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$?

Now apply the formula for determining the cross product of two vectors from their **i**-**j**-**k** components to determine $\mathbf{i} \times \mathbf{j}$.

6 **a** If $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ determine $\mathbf{a} \times \mathbf{b}$ and hence $|\mathbf{a} \times \mathbf{b}|$.

b Use the scalar product to determine, θ, the angle between the two vectors, and hence determine $|\mathbf{a} \times \mathbf{b}|$ using the fact that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

7 Determine a unit vector normal to the plane containing the vectors:

$$\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

8 The three points A, B and C, have position vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}-2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix}$ respectively.

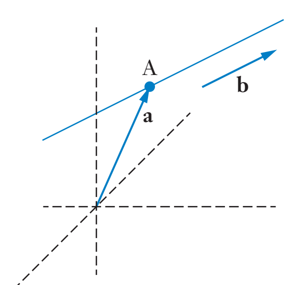
Find a unit vector perpendicular to the plane containing A, B and C.

Vector equation of a line

The vector equation of a line must be some rule that the position vector of all points on the line obey whilst all points not on the line do not obey. This can be done in three dimensions, just as it could in two dimensions, by the rule

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where **a** is the position vector of one point on the line,
b is a vector parallel to the line
and λ is some scalar.



However, whilst in two dimensions we also had the scalar product form of the vector equation of a straight line, $\mathbf{r}\cdot\mathbf{n} = c\ (= \mathbf{a}\cdot\mathbf{n})$ as mentioned in the previous chapter, this is not the case in three dimensions. This is because, in three dimensions, there are many lines that pass through the point with position vector **a** and are perpendicular to vector **n**. These lines together form the plane perpendicular to **n** and containing the point with position vector **a**, as we will see when we consider the vector equation of a plane.

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EXAMPLE 6

A line passes through the point with position vector $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and is parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Find **a** the vector equation of the line,

b the parametric equations of the line.

Solution

a A line through A, position vector $\mathbf{a}$, parallel to $\mathbf{b}$, has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

Thus the given line has vector equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

b If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Thus the parametric equations are:

$$\begin{cases} x = 2 + \lambda \\ y = -1 + \lambda \\ z = 3 + \lambda \end{cases}$$

Note: Eliminating λ from the parametric equations give the set of equations:

$$x - 2 = y + 1 = z - 3.$$

To generalise, the vector equation $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} + \lambda(p\mathbf{i} + q\mathbf{j} + r\mathbf{k})$

will give: $$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}.$$

These are the cartesian equations of a line through (a, b, c) and parallel to the vector $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$. This is mentioned here for the sake of completeness. The set of cartesian equations for a line in three-dimensional space is not specifically mentioned in the syllabus for this unit.

EXAMPLE 7

Show that the lines L_1: $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and L_2: $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 14 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ intersect and find the position vector of this point of intersection.

Solution

If the lines intersect there must exist values of λ and μ for which

$$\begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

i.e. $7 + 2\lambda = \mu$, [1]

$3 + 4\lambda = -1$, [2]

and $-2 - \lambda = 14 - 3\mu$. [3]

Solving [1] and [2] gives $\lambda = -1$ and $\mu = 5$,

values which are consistent with equation [3]: $-2 + 1 = 14 - 15$

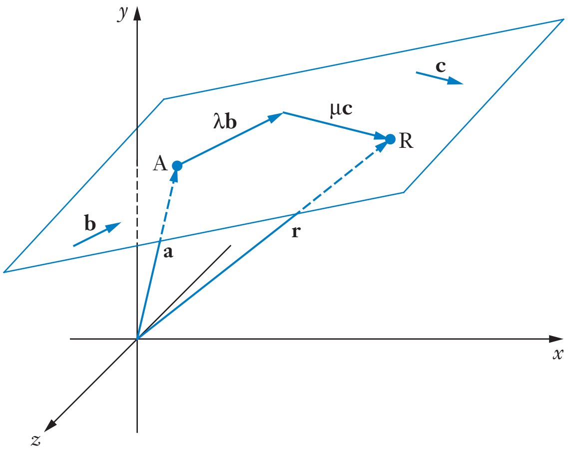
Hence L_1 and L_2 intersect at the point with position vector $\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$.

Vector equation of a plane

The vector equation of a plane needs to be a rule that the position vector of all points lying in the plane obey, and that all points not in the plane do not obey.

One way this can be achieved is to give the general position vector, **r**, in terms of **a**, the position vector of one point in the plane, and two other non-parallel vectors, **b** and **c**, that are parallel to the plane.

I.e. $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$



The plane containing a point with position vector **a** and parallel to the non-parallel vectors **b** and **c** has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.

For example, the plane containing point A, position vector $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and parallel to $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$

has equation: $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$.

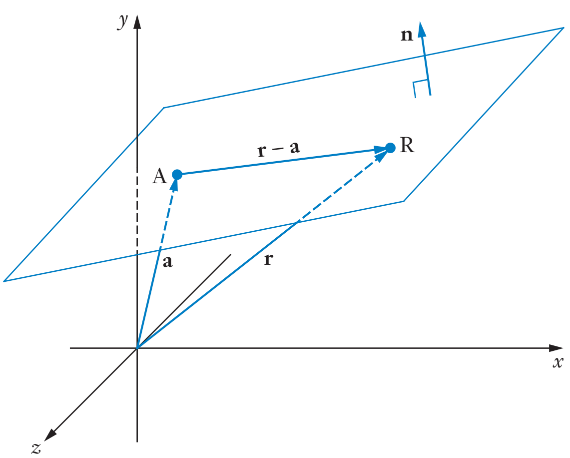
Alternatively the plane can be defined by giving **a**, the position vector of one point in the plane, and **n**, a vector that is perpendicular to the plane.
It then follows that

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

i.e. $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

i.e. $\mathbf{r} \cdot \mathbf{n} = c$

Thus, in three dimensions, $\mathbf{r} \cdot \mathbf{n} = c$ is the equation of a plane.



The plane containing a point with position vector **a** and perpendicular to the vector **n** has equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

Knowing **a** and **n** this becomes $\mathbf{r} \cdot \mathbf{n} = c$, where $c = \mathbf{a} \cdot \mathbf{n}$.

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For example, the plane containing point A, position vector $\begin{pmatrix}1\\4\\2\end{pmatrix}$ and perpendicular to $\begin{pmatrix}1\\-1\\3\end{pmatrix}$ has

the equation: $$\mathbf{r}\,.\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix}.\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

i.e. $$\mathbf{r}\,.\begin{pmatrix}1\\-1\\3\end{pmatrix} = 3$$

Writing $\mathbf{r}$ as $\begin{pmatrix}x\\y\\z\end{pmatrix}$ this becomes $$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\-1\\3\end{pmatrix} = 3$$

i.e. $$x - y + 3z = 3$$

This is the **cartesian equation** of the plane. Notice that the coefficients of x, y and z, i.e. (1, –1, 3) allow us to quickly determine a vector perpendicular to the plane, i.e. the vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

EXAMPLE 8

Line L has vector equation: $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

Plane Π has vector equation: $\mathbf{r}.(5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 14$

a Show that point A, position vector $\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, lies on L and in Π.

b Show that line L lies in the plane Π.

Solution

a If A lies on L there must exist some λ for which

$$\mathbf{i} + 5\mathbf{j} + \mathbf{k} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

i.e. $1 = 7 + 3\lambda$, $(\lambda = -2)$, $5 = 3 - \lambda$, $(\lambda = -2)$ and $1 = 5 + 2\lambda$, $(\lambda = -2)$.

Hence such a value of λ does exist and so point A lies on L.

Also $$(\mathbf{i} + 5\mathbf{j} + \mathbf{k}).(5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 5 + 15 - 6 = 14$$

The position vector of A satisfies the equation of Π. Point A lies in Π.

b If two points on L lie in Π then the line must lie in the plane. We already know A is on the line and in the plane. With $\lambda = 0$ we have $7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, the position vector of another point on the line.

Also $$(7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}).(5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 14$$

Thus we have two points on L that lie in the plane Π. The line L lies in Π.

Alternatively
- show that $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ satisfies $\mathbf{r}.(5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 14$ for all λ

or
- show that L is also perpendicular to $5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, and hence, with one point known to be in common, L lies in Π.

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EXAMPLE 9

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, the vector equation of the plane containing the line with vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and the point $\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$.

Solution

The given line is parallel to $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and so the plane must be parallel to $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

Putting $\lambda = 0$ gives the point $\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$ as a point on the line, and hence in the plane. Thus

$\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$, i.e. $\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, must be a vector parallel to the plane.

The required equation can be written: $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

EXAMPLE 10

Find the position vector of the point where the line

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

meets the plane $\mathbf{r} \cdot (-3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1$.

Solution

The position vector of the point where the line meets the plane will 'fit' both the equation of the line and that of the plane. If this position vector is **a** then

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

and $\mathbf{a} \cdot (-3\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1$

Thus $\begin{pmatrix} 2 + \lambda \\ 3 + 4\lambda \\ -1 - 2\lambda \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = 1$

$\therefore$ $-6 - 3\lambda + 3 + 4\lambda - 1 - 2\lambda = 1$

Solving gives $\lambda = -5$

Hence $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} - 5(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

$= -3\mathbf{i} - 17\mathbf{j} + 9\mathbf{k}$

The line meets the plane at the point with position vector $-3\mathbf{i} - 17\mathbf{j} + 9\mathbf{k}$.

Note that whilst the matrix form of vector representation was used for a while in the working of the previous example the final answer was given in the format used in the question, i.e. in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

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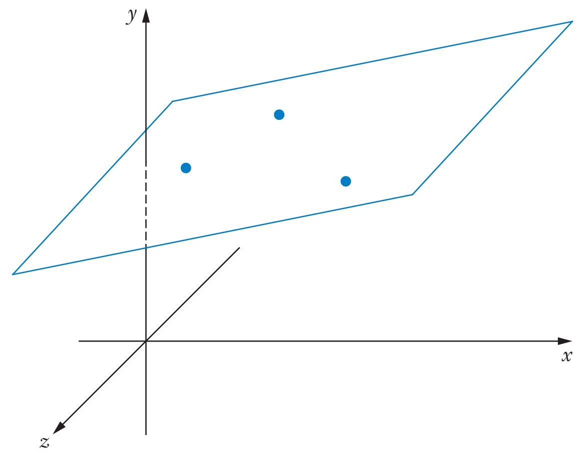
Given three points that lie in the plane

A plane can also be uniquely defined by giving three non-collinear points that lie in the plane. Hence, given three such points we should be able to determine the equation of the plane.



EXAMPLE 11

A plane contains three points, position vectors $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$, and $-9\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

Find the vector equation of the plane in the form

a $ax + by + cz = d$

b $\mathbf{r}.\mathbf{n} = c$

c $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

Solution

a The coordinates (–2, –2, 1), (10, –6, –3) and (–9, 5, 4) must each 'fit' the equation $ax + by + cz = d$.

Hence
$$\begin{cases} -2a - 2b + c = d \\ 10a - 6b - 3c = d \\ -9a + 5b + 4c = d \end{cases}$$

With the assistance of a calculator we solve for a, b and c to obtain each in terms of d:

$$a = \frac{2}{5}d, \quad b = -\frac{1}{5}d, \quad c = \frac{7}{5}d.$$

$$\left.\begin{cases} -2a-2b+c=d \\ 10a-6b-3c=d \\ -9a+5b+4c=d \end{cases}\right|_{a,b,c}$$

$$\left\{a = \frac{2\cdot d}{5}, b = \frac{-d}{5}, c = \frac{7\cdot d}{5}\right\}$$

Hence the equation is of the form

$$\frac{2}{5}dx - \frac{1}{5}dy + \frac{7}{5}dz = d.$$

Dividing by d and multiplying by 5 the equation can be written as

$$2x - y + 7z = 5$$

b From our answer to **a** the vector $2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ is perpendicular to the plane.

Hence the required equation is

$$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 5$$

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Alternatively we could use the cross product to determine a vector perpendicular to the plane, as shown below.

With points with position vectors $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $-9\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ lying in the plane it follows that

$$(10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) - (-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \quad \text{and} \quad (10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) - (-9\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$$

i.e. $\quad 12\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \quad$ and $\quad 19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}$

must be parallel to the plane.

Thus a vector perpendicular to the plane will be

$$\begin{aligned} & (3\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}) \\ = \; & -4\mathbf{i} + 2\mathbf{j} - 14\mathbf{k} \\ = \; & -2(2\mathbf{i} - \mathbf{j} + 7\mathbf{k}). \end{aligned}$$

i.e. $2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ is perpendicular to the plane.

$$\text{crossP}\left(\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 19 \\ -11 \\ -7 \end{bmatrix}\right) \qquad \begin{bmatrix} -4 \\ 2 \\ -14 \end{bmatrix}$$

The required equation is $\quad \mathbf{r}.(2\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = (-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).(2\mathbf{i} - \mathbf{j} + 7\mathbf{k})$

giving $\quad \mathbf{r}.(2\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 5, \quad$ as before.

c We determined above that the vectors $12\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ and $19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}$ are parallel to the plane.

Hence the required equation can be written

$$\mathbf{r} = -2\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(12\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \mu(19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k})$$

Note

- We could equally well have written the answer to part **c** as:

$$\mathbf{r} = 10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k} + \lambda(12\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \mu(19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k})$$

  or even (needs thought):

$$\mathbf{r} = 10\mathbf{i} - 6\mathbf{j} - 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} - \mathbf{k}) + \mu(-19\mathbf{i} + 11\mathbf{j} + 7\mathbf{k})$$

  or even (needs more thought):

$$\mathbf{r} = -9\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(7\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}) + \mu(-3\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

  Whilst these equations may appear different they all define the same plane, the plane with cartesian equation $2x - y + 7z = 5$.

- Justification that our answer to part **c**, i.e.

$$\mathbf{r} = -2\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda\,(12\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \mu\,(19\mathbf{i} - 11\mathbf{j} - 7\mathbf{k})$$

  is equivalent to the cartesian equation $2x - y + 7z = 5$ follows:

  Substituting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ leads us to the equations:

$$\begin{aligned} x &= -2 + 12\lambda + 19\mu & [1] \\ y &= -2 - 4\lambda - 11\mu & [2] \\ z &= 1 - 4\lambda - 7\mu & [3] \end{aligned}$$

  [2] – [3] gives $\quad y - z = -3 - 4\mu \quad$ i.e. $\quad \mu = \dfrac{-3 - y + z}{4}$

  [1] + 3 × [3] gives $\quad x + 3z = 1 - 2\mu \quad$ i.e. $\quad \mu = \dfrac{1 - x - 3z}{2}$

  Hence $\quad \dfrac{-3 - y + z}{4} = \dfrac{1 - x - 3z}{2}$

  which simplifies to $2x - y + 7z = 5$, as required.

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Inteception/collision

When we restricted our attention to two dimensions we found the vector equation of a line a useful approach when solving interception/collision questions.

When solving such questions we tended to replace λ in the equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ by t, for time.

This approach can also be used for these questions when three-dimensional space is involved, as the next example shows.

EXAMPLE 12

At time $t = 0$ the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/s) of two particles A and B are as given below:

$$\mathbf{r}_A = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} \quad \mathbf{v}_A = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{r}_B = \begin{pmatrix} -10 \\ 27 \\ -8 \end{pmatrix} \quad \mathbf{v}_B = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

Show that if the particles continue with these velocities they will collide and find the time of collision and the position vector of its location.

Solution

At time t seconds the position vectors of A and B will be $\mathbf{r}_A(t)$ and $\mathbf{r}_B(t)$ with:

$$\mathbf{r}_A(t) = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B(t) = \begin{pmatrix} -10 \\ 27 \\ -8 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}.$$

For collision to occur there must be some value of t for which

$$\begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 \\ 27 \\ -8 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}.$$

I.e. $5 + 2t = -10 + 3t$ Solving gives $t = 15$.

$-3 + t = 27 - t$ Solving gives $t = 15$.

and $7 + 4t = -8 + 5t$ Solving gives $t = 15$.

The particles will collide when $t = 15$ at the point with position vector $\begin{pmatrix} 35 \\ 12 \\ 67 \end{pmatrix}$.

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Exercise 5C

1 A line passes through the point with position vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and is parallel to $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Find **a** the vector equation of the line,

b the parametric equations of the line.

2 A line passes through point A, position vector $4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and point B, position vector $3\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Find **a** the vector equation of the line,

b the parametric equations of the line.

3 Write a vector equation for the plane that is perpendicular to the vector $3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and that contains the point A, position vector $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

4 Write a vector equation for the plane that contains the point A, position vector $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, and that is perpendicular to the vector $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$.

5 Write a vector equation for the plane that contains the point A, $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and that is parallel to the vectors $2\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$.

6 Write a vector equation for the plane that contains the point A, $\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$, and that is parallel to the vectors $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$.

7 The point with position vector $a\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$ lies on the line with vector equation

$$\mathbf{r} = 2\mathbf{i} + b\mathbf{j} - \mathbf{k} + \lambda(-3\mathbf{i} + \mathbf{j} + 2\mathbf{k}).$$

Determine the values of the constants a and b.

8 Write the cartesian equation of the plane with vector equation $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 21$.

9 State the vector equation $\mathbf{r} \cdot \mathbf{n} = c$ for the plane with cartesian equation

$$2x - 3y + 7z = 5$$

10 Prove that the line $\mathbf{r} = 2\mathbf{i} + 8\mathbf{j} - 3\mathbf{k} + \lambda(-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ is perpendicular to the plane $\mathbf{r} \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = c$.

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11 Show that the lines L_1: $\mathbf{r} = \begin{pmatrix} 10 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ and L_2: $\mathbf{r} = \begin{pmatrix} 0 \\ 8 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$ intersect and find the position vector of this point of intersection.

12 Show that the lines L_1: $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$
and L_2: $\mathbf{r} = 3\mathbf{i} + 13\mathbf{j} - 15\mathbf{k} + \mu(-\mathbf{i} + 4\mathbf{k})$
intersect and find the position vector of this point of intersection.

13 Line L_1 has equation: $\mathbf{r} = 13\mathbf{i} + \mathbf{j} + 8\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$.
Line L_2 has equation: $\mathbf{r} = 12\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \mu(5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k})$.
Line L_3 has equation: $\mathbf{r} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \beta(2\mathbf{i} + \mathbf{j} - \mathbf{k})$.

a Prove that lines L_1 and L_2 do not intersect.

b Prove that lines L_1 and L_3 intersect, find the position vector of the point of intersection and determine the angle between the lines.

14 Line L has vector equation: $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + \lambda(5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$
Plane Π has vector equation: $\mathbf{r}\cdot(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = 3$

a Show that point A, position vector $-4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$, lies on L.

b Show that point B, position vector $10\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, does not lie on L.

c Show that points A and B both lie in the plane Π.

d Show that line L lies in the plane Π.

15 At time $t = 0$ the position vectors ($\mathbf{r}$ m) and velocity vectors ($\mathbf{v}$ m/sec) of two particles A and B are as follows:

$$\mathbf{r}_A = \begin{pmatrix} -10 \\ 20 \\ -12 \end{pmatrix} \quad \mathbf{v}_A = \begin{pmatrix} 5 \\ -10 \\ 6 \end{pmatrix} \quad \mathbf{r}_B = \begin{pmatrix} -3 \\ -8 \\ 2 \end{pmatrix} \quad \mathbf{v}_B = \begin{pmatrix} 4 \\ -6 \\ 4 \end{pmatrix}$$

Show that if the particles continue with these velocities they will collide and find the time of collision and the position vector of its location.

16 Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, the vector equation of the plane containing the line

$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$ and the point $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Find the cartesian equation of this plane.

Hence express the equation of the plane in the form $\mathbf{r}\cdot\mathbf{n} = c$.

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17 Find the position vector of the point where the line

$$\mathbf{r} = 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$$

meets the plane $\mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 11$

18 Relative to a tracking station situated in space, the position vectors (**r** km) and velocity vectors (**v** km/h) of a spacecraft and of a piece of space debris at time $t = 0$ hours were as given below.

$\mathbf{r}_{\text{debris}} = 1200\mathbf{i} + 3000\mathbf{j} + 900\mathbf{k}$, $\quad \mathbf{v}_{\text{debris}} = 2000\mathbf{i} - 3600\mathbf{j} + 1000\mathbf{k}$.

$\mathbf{r}_{\text{spacecraft}} = 5750\mathbf{i} - 13\,250\mathbf{j} + 3370\mathbf{k}$, $\quad \mathbf{v}_{\text{spacecraft}} = 600\mathbf{i} + 1400\mathbf{j} + 240\mathbf{k}$.

Prove that if these velocities are maintained the spacecraft and the space debris will collide, and find the value of t for which this collision occurs.

19 A military fighter plane A wishes to intercept a supply plane B for mid-air refuelling. When the fighter pilot receives instructions to immediately change course and intercept B his position vector is $(80\mathbf{i} + 400\mathbf{j} + 3\mathbf{k})$ km. At that time B has position vector $(150\mathbf{i} + 470\mathbf{j} + 2\mathbf{k})$ km and is maintaining a constant velocity of $(300\mathbf{i} + 180\mathbf{j})$ km/h.

If the interception occurs 10 minutes later find the constant velocity maintained by the fighter during these ten minutes. (Ignore the final slow down necessary for smooth interception.)

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20 Plane Π_1 has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = 12.$

Plane Π_2 has equation $\mathbf{r}\cdot\begin{pmatrix}-2\\2\\-1\end{pmatrix} = 15.$

a Prove that Π_1 and Π_2 are parallel planes.

b Find the distance the planes are apart.

21 At time $t = 0$ (seconds) the position vectors (**r** m) and velocity vectors (**v** m/s) of two particles A and B are as follows:

$$\mathbf{r}_A = \begin{pmatrix}30\\-37\\-30\end{pmatrix} \quad \mathbf{v}_A = \begin{pmatrix}5\\8\\3\end{pmatrix} \quad \mathbf{r}_B = \begin{pmatrix}2\\40\\26\end{pmatrix} \quad \mathbf{v}_B = \begin{pmatrix}8\\0\\-2\end{pmatrix}$$

Assuming that the particles continue with these velocities find the minimum separation distance between the particles in the subsequent motion and the value of t for which it occurs.

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Vector equation of a sphere

If we extend our understanding of the equation of a circle in the x-y plane to three-dimensional space we obtain the equation of a sphere.

In three dimensions all points situated a distance a from some fixed point will form a sphere of radius a, centre at the fixed point.

The vector equation of a sphere centre (0, 0, 0) and radius a is:

$$|\mathbf{r}| = a.$$

Writing $\mathbf{r}$ as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we obtain the cartesian equation:

$$x^2 + y^2 + z^2 = a^2.$$

If the radius of the sphere is a and the centre has position vector $\mathbf{d}$ then the equation of the sphere is:

$$|\mathbf{r} - \mathbf{d}| = a.$$

Writing $\mathbf{r}$ as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{d}$ as $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ we obtain the cartesian equation:

$$(x - p)^2 + (y - q)^2 + (z - r)^2 = a^2.$$

EXAMPLE 13

Find the centre, radius and vector equation of the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = 6 - 2x + 4y + 10z.$$

Solution

Given $\quad x^2 + y^2 + z^2 = 6 - 2x + 4y + 10z$

i.e. $\quad x^2 + 2x + y^2 - 4y + z^2 - 10z = 6$

Create gaps: $\quad x^2 + 2x \quad + y^2 - 4y \quad + z^2 - 10z = 6$

Complete the squares: $\quad x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 - 10z + 25 = 6 + 1 + 4 + 25$

Hence $\quad (x + 1)^2 + (y - 2)^2 + (z - 5)^2 = 36$

The sphere has its centre at (–1, 2, 5),
a radius of 6 units
and vector equation $|\mathbf{r} - (-\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})| = 6$.

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EXAMPLE 14

Find the position vectors of the points where the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 9\mathbf{j})$ cuts the sphere $|\mathbf{r} - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})| = 7$.

Solution

If point A lies on the line and the sphere then $\mathbf{r}_A = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 9\mathbf{j})$

and $|\mathbf{r}_A - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})| = 7$

Thus $|\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 9\mathbf{j}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})| = 7$

i.e. $|2\lambda\mathbf{i} + (9\lambda - 3)\mathbf{j} + 3\mathbf{k}| = 7$

$\therefore$ $4\lambda^2 + 81\lambda^2 - 54\lambda + 9 + 9 = 49$

giving $85\lambda^2 - 54\lambda - 31 = 0$

Thus $\lambda = 1$ or $\lambda = -\frac{31}{85}$

Substituting these values into the equation of the given line gives the required position vectors as $3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$ and $\frac{1}{85}(23\mathbf{i} - 364\mathbf{j} + 170\mathbf{k})$.

Exercise 5D

Find the centre and radius of each of the following spheres.

1 $|\mathbf{r}| = 16$

2 $x^2 + y^2 + z^2 = 100$

3 $|\mathbf{r} - (\mathbf{i} + \mathbf{j} + \mathbf{k})| = 25$

4 $|\mathbf{r} - 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}| = 18$

5 $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 10$

6 $(x + 4)^2 + (y - 1)^2 + z^2 = 25$

7 $x^2 + y^2 - 8y + 16 + z^2 = 50$

8 $x^2 + y^2 + z^2 - 2x + 6y = 15$

9 $x^2 + y^2 + z^2 - 6y + 2z = 111$

10 $x^2 + y^2 + z^2 + 8x - 2y + 2z = 7$

For questions **11** to **18** state whether the given point lies inside, on or outside the sphere.

11 $|\mathbf{r}| = 5$, $(2, -3, 4)$.

12 $|\mathbf{r}| = 7$, $(-2, 3, 6)$.

13 $|\mathbf{r}| = 16$, $(7, 12, 9)$.

14 $|\mathbf{r} - (\mathbf{i} + \mathbf{j} - \mathbf{k})| = 8$, $(3, 1, 0)$.

15 $|\mathbf{r} - (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})| = 5$, $(3, 5, 2)$.

16 $|\mathbf{r} - (7\mathbf{i} + 10\mathbf{j} + 2\mathbf{k})| = 13$, $(2, -2, 2)$.

17 $(x - 1)^2 + (y + 3)^2 + (z - 2)^2 = 36$, $(5, -6, -1)$.

18 $x^2 + y^2 + z^2 - 4x - 3y - z = 61$, $(-1, 0, 8)$.

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19 A, B and C have position vectors $a\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$, $-4\mathbf{i} + b\mathbf{j} - 3\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + c\mathbf{k}$, respectively. All three points lie on the sphere $|\mathbf{r} - (\mathbf{i} + \mathbf{j} - 3\mathbf{k})| = 5\sqrt{2}$.

Find the values of a, b and c given that they are all positive constants.

20 Find the position vectors of points where the line $\mathbf{r} = \begin{pmatrix} -2 \\ 16 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$ cuts the sphere $\left| r - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right| = 5\sqrt{2}$.

21 Find the position vectors of points where the line $\mathbf{r} = 14\mathbf{i} - 9\mathbf{k} + \lambda(4\mathbf{i} + \mathbf{j} - 9\mathbf{k})$ cuts the sphere $|\mathbf{r} - (4\mathbf{i} + \mathbf{j} + 3\mathbf{k})| = 7$.

22 Prove that the line $\mathbf{r} = -2\mathbf{i} - \mathbf{j} - 11\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{k})$ touches but does not cut the sphere $|\mathbf{r} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})| = 5$. (i.e. Prove the line is a tangent to the sphere.)

Find the position vector of the point of contact.

23 Prove that the line $\mathbf{r} = \begin{pmatrix} 9 \\ 18 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix}$ is a tangent to the sphere $\left| \mathbf{r} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right| = 7$ and find the position vector of the point of contact.

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Distance from a point to a line revisited

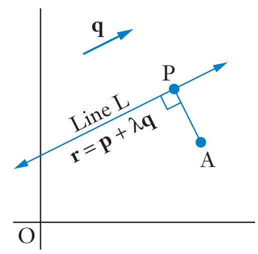
Note

This section uses the fact that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

At the time of writing, this application of the vector product is not specifically mentioned in the syllabus so it could be argued that such consideration is beyond the requirements of the unit. I include the use of the fact here to find the distance from a point to a line. The inclusion is for completeness, for interest, and in case that at some later stage it is made explicit that the inclusion of the cross product in the syllabus is to be taken as including the use of the above fact.

In the previous chapter, when considering vectors in the **i**-**j** plane, we used the scalar product to determine the distance from point A to a line L.

The method used the fact that $\mathbf{q} \cdot \overrightarrow{AP} = 0$. (See diagram, right.)



We can similarly use a scalar product approach to find the distance from a point to a line in three-dimensional space, as *Method one* of the next example shows. However, now that we have met the concept of a vector product of two vectors, and in particular that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, we could also use a vector product approach to find the distance from a point to a line, as *Method two* of the next example shows.

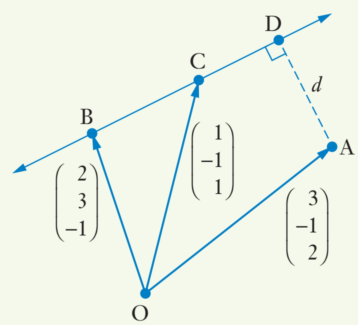
EXAMPLE 15

Find the distance from the point A, position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, to a line passing through points B and C, position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively.

Solution

Method one: A scalar product approach

A sketch of the situation is shown on the right. (This sketch does not need any accurate portrayal of relative positions. It simply allows us to formulate our method.)



We require the distance d shown in the diagram.

We will use the fact that $\overrightarrow{BC} \cdot \overrightarrow{AD} = 0$.

$$\overrightarrow{BC} = -\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AD} = -\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda\overrightarrow{BC}$$

$$= \begin{pmatrix} -1-\lambda \\ 4-4\lambda \\ -3+2\lambda \end{pmatrix}$$

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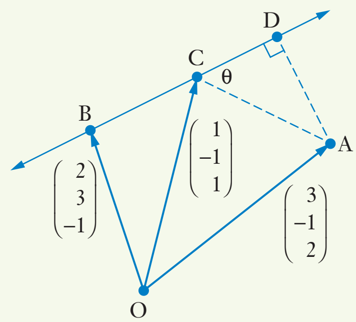
Thus $\begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1-\lambda \\ 4-4\lambda \\ -3+2\lambda \end{pmatrix} = 0$

Solving gives $\lambda = 1$

Thus $\overrightarrow{AD} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$

and hence $d = \sqrt{5}$ units of length.

Method two: A vector product approach



$$\begin{aligned}
\text{Required distance} &= |\overrightarrow{CA}| \sin\theta \\
&= (|\overrightarrow{BC}|\,|\overrightarrow{CA}| \sin\theta) \div |\overrightarrow{BC}| \\
&= |\overrightarrow{BC} \times \overrightarrow{CA}| \div |\overrightarrow{BC}|
\end{aligned}$$

Now $\overrightarrow{BC} = -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

and $\overrightarrow{CA} = 2\mathbf{i} + 0\mathbf{j} + \mathbf{k}$

Thus $\overrightarrow{BC} \times \overrightarrow{CA} = -4\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}$

$|\overrightarrow{BC} \times \overrightarrow{CA}| \div |\overrightarrow{BC}| = \sqrt{105} \div \sqrt{21}$

$= \sqrt{5}$, as in Method one.

Exercise 5E

For each of the following find the distance the given point is from the given line, in each case calculating the distance twice, once using a scalar product approach and once using a vector product approach.

1 Given point: Point A, position vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
Given line: Line through points B, position vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$,
and C, position vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$.

2 Given point: Point A, position vector $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
Given line: Line through points B, position vector $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$,
and C, position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

3 Given point: Point A, position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
Given line: Line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$.

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Miscellaneous exercise five

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Suppose $f(x) = 3x - 2$ and $g(x) = f(|x|)$.

Find **a** $f(3)$ **b** $f(-3)$ **c** $g(3)$

d $g(-3)$ **e** $f(5)$ **f** $g(-5)$

g Draw the graphs of $y = f(x)$ and $y = g(x)$.

2 AB is a diameter of a circle, lying in the **i**-**j** plane, with its centre at point P.

If point A has coordinates (1, 2) and B has coordinates (9, –4) find

a the coordinates of point P,

b the radius of the circle,

c the vector equation of the circle.

3 Find the radius and the cartesian coordinates of the centre of the following circles, each lying in the **i**-**j** plane.

a $|\mathbf{r} - (3\mathbf{i} - 2\mathbf{j})| = 7$

b $|\mathbf{r} - 2\mathbf{i} - 7\mathbf{j}| = 11$

c $(x - 3)^2 + (y + 2)^2 = 16.$

d $(x + 1)^2 + (y + 7)^2 = 20.$

e $x^2 + y^2 - 8x = 4y + 5.$

f $x^2 + 6x + y^2 - 14y = 42.$

4 Find to the nearest degree the acute angle between the lines L_1 and L_2 if L_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} - \mathbf{j})$ and L_2 has equation $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mu(2\mathbf{i} + 3\mathbf{j})$.

5 Find the range of each of the following for domain $\mathbb{R}$.

a $f(x) = x^2$ **b** $f(x) = x^2 + 3$ **c** $f(x) = (x + 3)^2$

d $f(x) = |x|$ **e** $f(x) = |x| + 3$ **f** $f(x) = |x + 3|$

6 What restriction is there on the possible values of a if

$$x^2 + 2x + y^2 - 10y + a = 0$$

is the equation of a circle?

7 If $f(x) = \dfrac{3}{x}$ and $g(x) = 2x - 1$ find the functions $f \circ g(x)$ and $g \circ f(x)$ in terms of x and state the natural domain and range of each.

8 Repeat the previous question but now for $f(x) = \sqrt{x + 3}$ and $g(x) = x^2 + 1$.

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9 If $z = 8\,\text{cis}\left(\frac{\pi}{6}\right)$, and $\bar{z}$ is the complex conjugate of z, determine each of the following, giving your answers in the form $r\,\text{cis}\,\theta$ for $r \geq 0$ and $-\pi < \theta \leq \pi$.

a $\bar{z}$ **b** $z + \bar{z}$ **c** $z - \bar{z}$ **d** $z\,\bar{z}$ **e** $z \div \bar{z}$

10 For $\{z: |z - (4 + 4\mathbf{i})| = 3\}$ determine

a the minimum possible value of $\text{Im}(z)$.

b the maximum possible value of $\text{Re}(z)$.

c the minimum possible value of $|z|$.

d the maximum possible value of $|z|$.

e the maximum possible value of $|\bar{z}|$.

11 (Without the assistance of a calculator.)

If $z = 3\,\text{cis}\left(\frac{5\pi}{6}\right)$ and $w = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ express each of the following in the form

$$r\,\text{cis}\,\theta \text{ for } r > 0 \text{ and } -\pi < \theta \leq \pi.$$

a $2z$ **b** $3w$ **c** zw **d** $\frac{z}{w}$ **e** iz

f $-w$ **g** $\bar{z}$ **h** $\overline{zw}$ **i** $\bar{z}\,\bar{w}$ **j** z^2w^3

12 (Without the assistance of a calculator.)

Display the three solutions to the equation $z^3 = \frac{27}{i}$ on an Argand diagram and express each in the form $r\,\text{cis}\,\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$.

13 (Without the assistance of a calculator.)

If $z = -1 + \sqrt{3}i$ and $\bar{z}$ is the complex conjugate of z express

$$\left(z + \frac{1}{\bar{z}}\right)^4 \text{ and } \left(z - \frac{1}{\bar{z}}\right)^4$$

in the form $r\,\text{cis}\,\theta$ for $-\pi \leq \theta \leq \pi$.

14 For each of the following conditions show diagrammatically the set of all points lying in the complex plane and obeying the condition.

a $z + \bar{z} = 4$ **b** $|z - i| = 2$ **c** $0 \leq \arg(z - 2) \leq \frac{2\pi}{3}$

15 The line $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \lambda(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ passes through A (where $\lambda = 1$) and B (where $\lambda = 5$). Find the position vector of point C where $\overrightarrow{AC} : \overrightarrow{BA} = -1 : 4$.

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16 Each part of this question gives the vector equations of two lines.

For each part determine whether the lines are parallel lines,
or intersecting lines,
or skew lines.

a $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$
and $\mathbf{r} = 5\mathbf{j} + 2\mathbf{k} + \mu(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

b $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 10\mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$
and $\mathbf{r} = -5\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

c $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(-\mathbf{i} + \mathbf{j} + \mathbf{k})$
and $\mathbf{r} = 2\mathbf{j} - 7\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{k})$

d $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$
and $\mathbf{r} = 5\mathbf{i} + 6\mathbf{j} + \mu(-\mathbf{i} - \mathbf{j} + \mathbf{k})$

17 At time $t = 0$ seconds the position vectors, $\mathbf{r}$, and velocity vectors, $\mathbf{v}$, of a tanker and a submarine are as follows. (The $\mathbf{i}$–$\mathbf{j}$ plane is the surface of the sea.)

$$\begin{aligned}\mathbf{r}_{\text{Tanker}} &= (1150\mathbf{i} + 827\mathbf{j})\text{ m}\\ \mathbf{r}_{\text{Sub}} &= (1345\mathbf{i} + 970\mathbf{j})\text{ m}\\ \mathbf{v}_{\text{Tanker}} &= (10\mathbf{i} - 2\mathbf{j})\text{ m/s}\\ \mathbf{v}_{\text{Sub}} &= (-5\mathbf{i} - 13\mathbf{j} - 4\mathbf{k})\text{ m/s}\end{aligned}$$

If both vessels maintain these velocities, show that the tanker passes directly over the submarine, find the value of t when this occurs and find the depth of the submarine at the time.

18 Use vector methods to prove that in the parallelogram OABC the line drawn from O to the mid-point of AB cuts AC at the point of trisection of AC that is nearer to A.

19 A defensive missile battery launches a ground-to-air missile A to intercept an incoming enemy missile B. At the moment of A's launch the position vectors of A and B (in metres), relative to the defensive command headquarters were:

$$\mathbf{r}_A = \begin{pmatrix}600\\0\\0\end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix}2200\\4000\\600\end{pmatrix}$$

A and B maintain the velocities (in m/s): $\mathbf{v}_A = \begin{pmatrix}-196\\213\\18\end{pmatrix}$ and $\mathbf{v}_B = \begin{pmatrix}-240\\100\\0\end{pmatrix}$

Prove that A will *not* intercept B and find 'how much it misses by'.

Suppose instead that the computer on missile A detects that it is off target and, 20 seconds into its flight, A changes its velocity and interception occurs after a further 15 seconds. Find, in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the constant velocity that A must maintain during this final 15 seconds for the interception to occur.

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